Outline:

I. Weil - Siegel - Mass Formula + consequences.

II. Adelic Interpretation

III. Tamagawa measure, Weil's Conj, example of SL_n.

IV. Tamagawa # in Number Theory (History: Langlands, Lai, Kottwitz; Tamagawa # of Motives)

V. Weil Conjecture for Function Fields.

VI. Example of $\mathbb{P}^2$.

VII. Cohomological Reinterpretation.

VIII Topological Inspiration

I. Smith - Minkowski - Siegel Formula

Let $q \in R[x_1, \dots, x_n]$ be homogeneous of degree 2.

If $M \in GL_n(R)$ we may transform q:

$$q\left(M^{-1}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) \stackrel{\Delta}{=} q'\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

we call q and q' equivalent.

Question: given R and n, can we determine all equivalence classes of quadratic forms?

Answer varies greatly based upon R.

If R is a field, char $\neq 2$, the form can always be diagonalized;

$$\lambda_1 x_1^2 + \dots + \lambda_n x_n^2 \qquad \lambda_i \in R. \quad (\text{These are not unique}).$$

If $R = \mathbb{C}$, we may always write as $x_1^2 + \dots + x_r^2$ (r = rank)

If $R = \mathbb{R}$, it is equivalent to $x_1^2 + \dots + x_a^2 - x_{a+1}^2 - \dots - x_{a+b}^2$

for a unique a and b.

If $R = \mathbb{Q}_p$: determined by an elt. of $\mathbb{Q}_p^\times / \mathbb{Q}_p^{\times 2}$ ("discriminant") and an elt. in $\{\pm 1\}$ ("Hasse Invariant").

p is odd: 8 iso classes

p is 2: 16 iso classes

If $R = \mathbb{Z}$, $n = 2$ ... this is almost $1/2$ of Gauss's Disquisitiones Arithmeticae. "(Integral) Binary Quadratic Forms."

Gauss had a biological view of quadratic forms;

he investigated "families" of quadratic forms that were "coarser" than equivalence. One such family is the **genus**, which we will come to below. These days we view genera as a (Hasse) local-global principle.

Let $R = \mathbb{Q}$ for now. Observe that q can be naturally "base-changed" to $\mathbb{Q}_p$, or $\mathbb{R}$. There detection of equivalence is easy; see above.

On the other hand:

Thm. (The Hasse Principle). Let q and q' be quadratic forms over $\mathbb{Q}$. q and q' are iso iff:

$$q_{\mathbb{R}} \simeq q'_{\mathbb{R}}$$

$$q_{\mathbb{Q}_p} \simeq q'_{\mathbb{Q}_p} \quad \forall p. \qquad (\simeq = \text{"is equivalent to"}).$$

I.e. equivalent locally $\Rightarrow$ equivalent globally for quadratic forms.

What about over $\mathbb{Z}$?

Def. Let q, q' be quadratic forms $/\mathbb{Z}$. We say q, q' are of the same genus if:

a) $q_{\mathbb{R}} \simeq q'_{\mathbb{R}}$

b) $\forall N$, $q_{\mathbb{Z}/N\mathbb{Z}} \simeq q'_{\mathbb{Z}/N\mathbb{Z}}$.

Def. A quadratic form $q_{\mathbb{Z}}$ is <u>positive</u> <u>definite</u> if

$q_{\mathbb{R}}$ is.

Equivalent $\Longrightarrow$ Same genus, but NOT conversely. However, there are only a finite number of positive definite quadratic forms in each genus.

Notation: let $\Lambda = \mathbb{Z}^n$, $q : \Lambda \longrightarrow \mathbb{Z}$. The function:

$$b(x,y) \overset{\Delta}{=} q(x+y) - q(x) - q(y)$$

determines a bilinear form. This in turn gives

$$\Lambda \xrightarrow{e} \Lambda^{\vee} = \mathrm{Hom}(\Lambda, \mathbb{Z}).$$

$$b \longmapsto b(x, *)$$

q is positive def $\Rightarrow$ e is injective.

Def. Because $b(x,x) = 2q(x)$ is even, we call $(\Lambda, q/_{\mathbb{Z}})$ an <u>even</u> lattice.

<u>Def.</u> Let $O_q(\mathbb{Z}) \overset{\Delta}{=}$ Automorphisms of Λ fixing $q|_{\Lambda}$.

($O_q(\mathbb{Z})$ is finite b.c. q is pos-def).

<u>Thm.</u> (Smith-Minkowski-Siegel Mass Formula). Let $\Lambda = \mathbb{Z}^n$, $q : \Lambda \longrightarrow \mathbb{Z}$ a pos-def quadratic form.

Then

$$\sum_{q' \in \mathrm{Genus}(q)} \frac{1}{|O_{q'}(\mathbb{Z})|} = \frac{2\,|\Lambda^{\vee}/\Lambda|^{(n+1)/2}}{\prod_{m=1}^{n} \mathrm{vol}(S^{m-1})} \prod_p c_p$$

where $\mathrm{Vol}(S^{m-1}) = \dfrac{2\pi^{m/2}}{\Gamma(m/2)}$ = volume of standard $(m-1)$-sphere;

$$c_p \text{ (for each prime } p) = \frac{2\,p^{kn(n-1)/2}}{|O_q(\mathbb{Z}/p^k\mathbb{Z})|} \quad \text{for } k >> 0.$$

Note: we may pick $k = 1$ if $p \nmid |\Lambda^{\vee}/\Lambda|$. $p \mid |\Lambda^{\vee}/\Lambda|$ are the "bad" primes.

Note: If $\Lambda^\vee = \Lambda$, we call this (q, Λ) "unimodular". Such lattices necessarily have dim. div by 8. For fixed n, $8 \mid n$, (q, q' unimodular) $\Leftrightarrow$ (q is in same genus as q'). In this case we may group together terms in the SMS formula.

<u>Thm.</u> Let $8 \mid n$; $(q, \Lambda \triangleq \mathbb{Z}^n)$ a unimodular quadratic lattice. Then:

$$\sum_q \frac{1}{|O_q(\mathbb{Z})|} = \frac{2\, \zeta(2)\, \zeta(4) \cdots \zeta(n-4)\, \zeta(n-2)}{\mathrm{vol}(S^0) \cdots \mathrm{vol}(S^{n-1})}\, \zeta(n/2)$$

$$= \frac{B_{n/4}}{n} \prod_{1 \le j < n/2} \frac{B_j}{4j}.$$

This number is called the "mass" of the genus: it counts each equivalent lattice inversely to its automorphisms.

For $n = 0$, it is 1.

$n = 8$, it is $\dfrac{1}{696729600} = \dfrac{1}{|\mathrm{Aut\ of\ } E_8|}$

$\Rightarrow$ E_8 is the unique unimodular pos-def even lattice.

$n = 16$, Two even unimodular lattices,

$$\frac{1}{|\mathrm{Aut}\, E_8|^2} + \frac{1}{2^{15} \cdot 16} = \frac{691}{277{,}667{,}181{,}515{,}243{,}520{,}000}$$

(quintillion, quadrillion, trillion, billion, million, Thous)

$n = 24$: 24 even unimodular lattices, call Niemeier Lattices. (One is Leech lattice).

$n = 32$: Must be > 80 million such lattices.

History: The SMS formula, in 2-dimensions, is equiv. to Dirichlet's class # for imag. quadratic Fields (1839)

In dim 3, Eisenstein gave partial results.

1867: H.J.S. Smith discovered the higher dimensional formula; they were forgotten

1885: Minkowski rediscovered; w/ errors

1935: Siegel corrects the errors.

1964, 1965: Weil extends this to the Siegel–Weil formula which gives the SMS formula as a constant term (building on Siegel 1951, 1952).

II. Enter the Adèles. "Adeles make life possible" — J. Arthur.

Two quadratic forms q and q' are in the same genus if they are equivalent mod N $\forall N$, and over $\mathbb{R}$. This is equivalent to

$$q_{\hat{\mathbb{Z}}\times\mathbb{R}} \xrightarrow{\alpha} q'_{\hat{\mathbb{Z}}\times\mathbb{R}}, \quad \hat{\mathbb{Z}} = \varprojlim \mathbb{Z}/N\mathbb{Z}$$

Meanwhile, by the Hasse principle we know that

$$q_{\mathbb{Q}} \xrightarrow{\beta} q'_{\mathbb{Q}}.$$

Thus we find that $\beta^{-1}\alpha$ is a well-defined elt. of

$$O_q(\mathbb{Q}) \backslash O_q(\mathbb{A}) / O_q(\hat{\mathbb{Z}}\times\mathbb{R})$$

where $\mathbb{A}_{\mathbb{Q}} = \prod_{v \text{ places}} \mathbb{Q}_v = \mathbb{Q}\cdot(\hat{\mathbb{Z}}\times\mathbb{R})$ is the ring of adèles.

Claim. This double quotient is in isomorphism w/ $\{(q',\Lambda) \mid \Lambda \cong \mathbb{Z}^n,\ q' \text{ same genus as } q\}$

as groupoids.

Thm : $\displaystyle\sum_{\substack{q' \text{ in} \\ \text{same genus} \\ \text{as } q}} \frac{1}{|O_{q'}(\mathbb{Z})|}$ is the "mass" of the Adelic double quotient. This sum is trivially equivalent to

$$\sum_{\gamma}' \frac{1}{|O_q(\hat{\mathbb{Z}}) \cap \gamma^{-1} O_q(\mathbb{Q}) \gamma|}, \quad \gamma \in O_q(\mathbb{Q}) \backslash O_q(\mathbb{A}) / O_q(\hat{\mathbb{Z}} \times \mathbb{R}) \simeq O_q(\mathbb{Q}) \backslash O_q(\mathbb{A}_f) / O_q(\hat{\mathbb{Z}})$$

(f = finite Adeles, discarding Archimedean places.)

Rmk. Recall: if $\mathfrak{X}$ is an Artin stack, $\operatorname{spec} \mathbb{F}_q \xrightarrow{x} \mathfrak{X}$ a map, then $|x| = \frac{1}{|\operatorname{Aut} \mathfrak{X}_x|}$. This way $|X/_G(\mathbb{F}_q)| \triangleq \sum_{x : \operatorname{spec} \mathbb{F}_q \to X/_G} |x|$ is exactly $|X(\mathbb{F}_q)| / |G(\mathbb{F}_q)|$.

It is an amazing fact that $O_q(\mathbb{Q}) \subseteq G_q(\mathbb{A})$ is discrete and cocompact. Let μ be a left Haar measure on $G_q(\mathbb{A})$.

Claim.
$$\sum_{q' = q}' \frac{1}{|O_{q'}(\mathbb{Z})|} = \frac{\mu(O_q(\mathbb{Q}) \backslash O_q(\mathbb{A}))}{\mu(O_q(\hat{\mathbb{Z}} \times \mathbb{R}))}$$

Indeed: the RHS, expanded place-by-place, is precisely the RHS of the SMS Mass Formula.

So we see a relationship:

"Stacky" point count $\Longleftrightarrow$ Volume of Adelic Quotients

III. Weil's Tamagawa Number Conjecture.

Let G be a semisimple algebraic group $/K$, a global field. Choose, for each completion

K_v at a place v, the Haar measure s.t. G_v has volume 1 almost everywhere, and such that the induced measure on $\mathbb{A}/K$ is 1.

Now: Say $n = \dim G$.

Let ω = a left-invariant n-form on $G(K)$.

This gives a top-form on $G(K_v)$ for completions at any place v of K. Together w/ the measures on K_v, this induces a Haar measure on $G(K_v)$ for all v. Multiplying these gives a measure on $G(\mathbb{A})$. This is called the Tamagawa measure. Choice of a <u>different</u> top form changes the overall measure by a factor of 1 by the product formula:

$\prod_v |\lambda|_v = 1$ for $\lambda \in K^\times$, so the Tamagawa measure is well-defined.

It is a wonderful theorem that $G(K) \subseteq G(\mathbb{A}_K)$ is discrete.

<u>Def.</u> The Tamagawa Number of a group G, $\tau(G)$ is defined as $\mu_{Tam}(G(K) \backslash G(\mathbb{A}_K))$.

I.e. the canonical measure of Automorphic space!

<u>Conj / Thm.</u> (Weil's Tamagawa # conjecture).

Let G be a <u>simply connected</u> (i.e. w/out proper algebraic covering) semisimple algebraic group.

Then $\tau(G) = 1$.

Ex. $SL_n = G$, $K = \mathbb{Q}$.

We must compute:

$$SL_n(\mathbb{Q}) \backslash SL_n(\mathbb{A}_{\mathbb{Q}})$$

w.r.t. the Tamagawa measure. Choose an iso $\det_{\mathbb{Q}}(\mathfrak{sl}_n(\mathbb{Q})) \cong \mathbb{Q}$.

This gives us measures μ_p on $SL(\mathbb{Q}_p)$ for all $p \leq \infty$. Now, the strong approximation theorem gives us:

$$\mu(SL_n(\mathbb{Q}) \backslash SL_n(\mathbb{A}_{\mathbb{Q}})) = \mu_\infty(SL_n(\mathbb{Z}) \backslash SL_n(\mathbb{R})) \prod_{p<\infty} \mu_p(SL_n(\mathbb{Z}_p)).$$

Now, if we choose the above iso so that

$\det_{\mathbb{Z}}(\mathfrak{sl}_n \mathbb{Z}) \cong \mathbb{Z}$, we have:

$$[\mathfrak{sl}_n \mathbb{Z}_p : p\,\mathfrak{sl}_n \mathbb{Z}_p] = p^{n^2-1}.$$

$$\mathrm{Exp}(p\,\mathfrak{sl}_n \mathbb{Z}_p) = \Gamma_1(\mathbb{Z}_p) \implies \mu_p(\Gamma_1(\mathbb{Z}_p)) = p^{1-n^2}$$

But $[SL_n(\mathbb{Z}_p) : \Gamma_1(\mathbb{Z}_p)] = (p^n - p^{n-1})(p^n - p^{n-2}) \dots (p^n - 1)/_{p-1}$

$$\frac{(1-1/p)\cdot p}{(p-1)} = 1$$

$$\Rightarrow \mu_p(SL_n(\mathbb{Z}_p)) = \frac{(p^n - p^{n-1}) \dots (p^n - 1)}{(p-1) \cdot p^{n^2}} \cdot p = \prod_{i=2}^{n} (1 - p^{-i}).$$

Thus

$$\tau(G) = 1 \iff \prod_{i=2}^{n} \zeta(i) = \mu_\infty(SL_n(\mathbb{Z}) \backslash SL_n(\mathbb{R}))$$

⇓ Special values of ζ, including

⇓ Volume of a fundamental domain.

odd values!

We see that the Weil Tamagawa Number Conjecture masks great complexity!

IV. Tamagawa In Number Theory.

The progress on Weil's conjecture went as follows:

Weil: 1959: Computed τ for several classical groups.

Ono: 1963 Computes the Tamagawa Number for algebraic tori. Here it could be rational as opposed to integral and depends upon Galois Cohomology.

Langlands: 1966. Proved Weil's conjecture for split Chevalley groups

Lai: 1980 (Langlands's student). Proved Weil's conjecture for quasisplit reductive groups

Kottwitz: 1980. Proved for all groups satisfying the Hasse Principle — all groups except w/ E_8 factors

Chernousov: 1989. Proved Hasse Principle for E_8.

2011: Gaitsgory and Lurie announce proof in the function field case.

Langlands: let $\mathfrak{g}_{\mathbb{Q}}$ be semisimple Lie Algebra $\subseteq \mathrm{End}(V_{\mathbb{Q}})$. $\mathfrak{h}_{\mathbb{Q}}$ Cartan subalgebra (split). $X_\alpha, X_{-\alpha} \in \mathfrak{g}_{\pm\alpha}$ s.t. $[X_\alpha, X_{-\alpha}] = H_\alpha$, $\alpha(H_\alpha) = 2$, $\exists$ automorphism θ of $\mathfrak{g}_{\mathbb{Q}}$ s.t. $\theta(X_\alpha) = X_{-\alpha}$. Let $M \subseteq V_{\mathbb{Q}}$ be some lattice s.t.

i) $M = \sum_{\lambda \in \text{weights of } \mathfrak{h}_{\mathbb{Q}}} M \cap V(\lambda)$

ii) $(X_\alpha^n / n!) M \subseteq M \quad \forall \alpha$.

Let $G_{\mathbb{C}}$ = Lie group w/ Lie alg. $\mathfrak{g}_{\mathbb{C}}$ that is contained in $\mathrm{End}_{\mathbb{C}}(V_{\mathbb{C}})$ $(V_{\mathbb{C}} = V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C})$.

Let $G_{\mathbb{Z}} = \{ g \in G_{\mathbb{Q}} \mid gM = M \}$.

Let ω be a top form on $G_{\mathbb{R}}$ s.t.

$$\bigwedge H_i \wedge \bigwedge X_\alpha \longmapsto \pm 1.$$

Let $[dg]$ = corresponding Haar measure.

Let $\prod_{i=1}^{p} (t^{2a_i - 1} + 1)$ be the Poincaré polynomial of $G_{\mathbb{C}}$, $c = |\pi_1(G_{\mathbb{C}})|$. Then:

$$\int_{G_{\mathbb{Z}} \backslash G_{\mathbb{R}}} [dg] = c \prod_{i=1}^{p} \zeta(a_i),$$

ζ = Riemann Zeta Function.

I.e. Tamagawa Number = 1 $\Longleftrightarrow$

Archimedean volume = special value of ζ associated to topological invariants of G.

Finally, we note that for a "motive" M (a suitable "direct factor" of a universal cohomology theory $H^r(X, \mathbb{Z}(n))$ for smooth complete varieties X), Bloch–Kato postulate abelian groups

$$A(\mathbb{Q}), \quad A(\mathbb{Q}_p) \quad \text{and a map} \quad A(\mathbb{Q}) \to A(\mathbb{Q}_p)$$

including primes of bad reduction and archimedean places. [One hopes that morally $A(\mathbb{Q}) \simeq H^{r+1}_{\mathcal{M}}(X, \mathbb{Z}(n))$ at least away from bad fibers]. Then

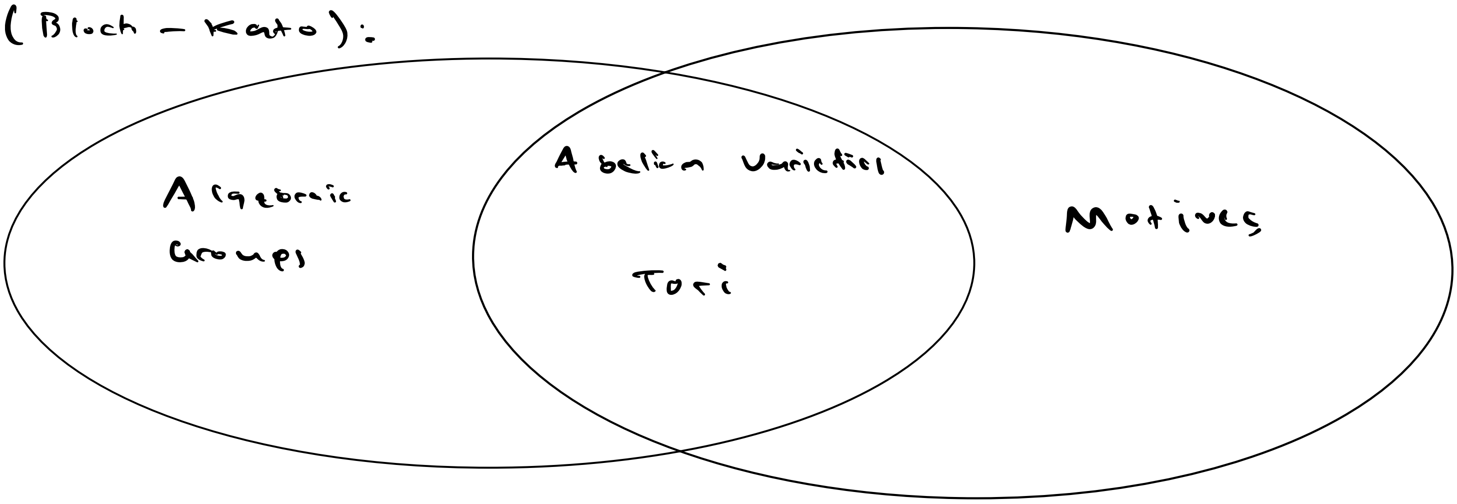
$$\operatorname{Tam}(M) \overset{?}{=} \mu\left(\prod_{p \leq \infty} A(\mathbb{Q}_p) \Big/ A(\mathbb{Q}) \right).$$

Conjecture (Bloch–Kato).

$$\mathrm{Tam}(M) = \# H^0\left(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), M^* \otimes \mathbb{Q}/\mathbb{Z}(1)\right) \cdot \left(Ш(M)\right)^{-1}$$

where $Ш(M)$ is a (conjecturally finite) group defined in BK. Generalizes Ono's Tamagawa # formula for tori.

The land of Tamagawa Numbers (Bloch–Kato):



Is there a larger oval containing both?

V. Weil's conjecture for function fields

Firstly, we recall that, for K a function field in one variable of a smooth projective curve $X/\mathbb{F}_q$,

$$G(K) \backslash G(\mathbb{A}) / G(\mathbb{O}) \simeq \mathrm{Bun}_G(X).$$

Thus we hope to use the geometry of Bun_G to derive information about the Tamagawa number.

Let $G \xrightarrow{\pi} X$ be a relative group scheme, $\mathcal{L}$ denote the pullback of $\Omega^d_{G/X}$ along the identity section

$e: X \to G$; let $\omega \in \mathcal{L}$ be a rational section.

Local computations show us (as we saw w/ $SL_n(\mathbb{Q})$):

$$\mu_{x,\omega}(G(m_x)) = q^{-\deg x \, v_x(\omega)} \frac{1}{|K(x)|^d}$$

$$\Rightarrow \mu_{x,\omega}(G(\mathcal{O}_x)) = q^{-\deg x \, v_x(\omega)} \frac{G(K(x))}{|K(x)|^d}.$$

Def. Let G_0 be a connected semisimple algebraic group $/K(X)$. Define the Tamagawa measure as the product

$$\mu_{Tam} = q^{d(1-g)} \prod_{x \in X} \mu_{x,\omega}.$$

I.e. $\mu_{Tam}(G(\mathbb{A}_0)) = q^{d(1-g) - \deg \mathcal{L}} \prod_{x \in X} \frac{|G(K(x))|}{|K(x)|^d}$, $\mathbb{A}_0$ = integral Adeles

Because G_0 is semisimple, left-invariant top form is also right-invariant

Thus μ_{Tam} descends to the quotient $G(K) \backslash G(\mathbb{A})$

$$\int_{x \in G(\mathbb{A})} f(x)\, d\mu_{Tam} = \int_{y \in G(K)\backslash G(\mathbb{A})} \Big(\sum_{\pi(x)=y} f(x) \Big) d\mu_{Tam}$$

The Tamagawa Number of G_0 is defined to be

$$\mu_{Tam}(G(K) \backslash G(\mathbb{A})).$$

Ex. $G = G_a$. $\dim d = 1$, $\mathcal{L} \cong \mathcal{O}_X$. $|G(K(x))| = |K(x)|$ $\forall x \in X$. Thus $\mu_{Tam}(G(\mathbb{A}_0)) = q^{1-g}$. Now the Weil exact sequence becomes:

$$0 \to H^0(X, \mathcal{O}_X) \to G(\mathbb{A}_0) \to G(K) \backslash G(\mathbb{A}) \to H^1(X, \mathcal{O}_X) \to 0$$

$$\text{Thus } \tau(G_a) = \frac{|H^1(X, \mathcal{O}_X)|}{|H^0(X, \mathcal{O}_X)|} \mu_{Tam}(G(\mathbb{A}_0)) = \frac{q^g}{q} q^{1-g} = 1.$$

This explains the factor q^{1-g}: we want $\mathrm{Vol}(\mathbb{A}/K)$ $= \mu(G_a(K) \backslash G_a(\mathbb{A})) = 1$ as in # field case. Moreover, the q^{1-g} term cause τ to behave well under Weil restriction.

Conjecture. Let G be semi-simple, simply connected. Then:

$$\mu_{Tam}(G(K)\backslash G(\mathbb{A})) = 1.$$

Let us unpack this:

$$\mu_{Tam}(G(K)\backslash G(\mathbb{A})) = \sum_{\gamma} \frac{\mu_{Tam}(G(\mathbb{A}_0))}{|G(\mathbb{A}_0) \cap \gamma^{-1} G(K) \gamma|}$$

$$= q^{d(1-g) - \deg \mathcal{L}} \sum_{\gamma} \frac{1}{|G(\mathbb{A}_0) \cap \gamma^{-1} G(K)\gamma|}$$

I.e.: $\displaystyle\prod_{x \in X} \frac{|k(x)|^d}{|G(k(x))|} = q^{d(1-g) - \deg(\mathcal{L})} \sum_{\gamma} \frac{1}{|G(\mathbb{A}_0) \cap \gamma^{-1} G(K)\gamma|}$

where $\gamma \in G(K)\backslash G(\mathbb{A}) / G(\mathbb{A}_0)$.

But the RHS is simply the stacky mass of Bun_G!

So we get:

<u>Conj.</u> $$\prod_{x \in X} \frac{|k(x)|^d}{|G(k(x))|} = q^{d(1-g) - \deg(\mathcal{L})} \sum_{\mathcal{P} \in Bun_G(X)(\mathbb{F}_q)} \frac{1}{|Aut(\mathcal{P})|}.$$

This is our analogue of the SMS formula.

Note that a priori neither side converges! This convergence is part of the theorem. On the right hand side, it is because the deeper layers of the HN filtration grow very quickly; on the LHS convergence is related to the Weil conjectures.

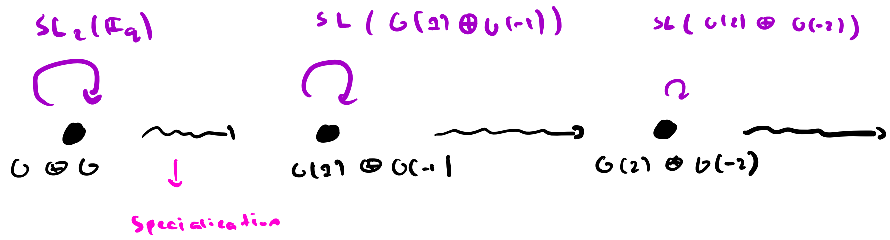
VI. Example of $Bun_{SL_2}\mathbb{P}^1$

Let $G = SL_2$, $X = \mathbb{P}^1$.

$$|X(\mathbb{F}_q)| = q+1$$

$$|G(\mathbb{F}_q)| = \frac{(q^2-1)(q^2-q)}{q-1} = (q+1)q(q-1) \qquad d = 3.$$

$\mathrm{Bun}_{SL_2}(\mathbb{P}^1)$ looks like:



$$|SL(\mathcal{O}(1)\oplus\mathcal{O}(-1))| = q^3\cdot(q-1).$$

$$|SL(\mathcal{O}(2)\oplus\mathcal{O}(-2))| = q^5\cdot(q-1).$$

$\vdots$

$\mathbb{A}^1 \cup \{\infty\}$ ↗ $\mathbb{F}_q$

↓

$q,\ q^2-q,\ q^3-q,\ q^4-q^2,\ q^5-q$

$q^6-q^3-q^2+q.$

LHS:
$$\prod_x \frac{|K(x)|^d}{|G(K(x))|}$$

($\mathbb{A}^1 \cup \{\infty\}$.)

$$= \left[\frac{(q^2)^3}{(q+1)q(q-1)}\right]^{q+1} \cdot \left[\frac{(q^2)^3}{(q^2+1)(q^2)(q^2-1)}\right]^{(q^2-q)/2}$$

$$\cdot \left[\frac{(q^3)^3}{(q^3+1)q^3(q^3-1)}\right]^{(q^3-q)/3} \cdots \left[\frac{(q^n)^3}{(q^n+1)q^n(q^n-1)}\right]^{\frac{1}{n}\sum_{r|n}\mu(n/r)\cdot q^r} \cdots$$

RHS:
$$q^3\left(\frac{1}{(q+1)q(q-1)} + \frac{1}{(q-1)}\left[\frac{1}{q^3}+\frac{1}{q^5}+\cdots\right]\right)$$

$$= q^3\left(\frac{1}{(q+1)q(q-1)} + \frac{1}{(q-1)q^3(1-1/q^2)}\right)$$

$$= \frac{q^3}{(q-1)^2(q+1)}.$$

Not clear why/how these are equal.

VII. Cohomological Reinterpretation.

Recall that $\mathrm{Bun}_G(X)$ is smooth over $\mathbb{F}_q$, and its dimension is precisely $d(g-1) - \deg \mathcal{L}$. Then we see

<u>Conj.</u>

$$\frac{|\mathrm{Bun}_G(X)(\mathbb{F}_q)|}{q^{\dim(\mathrm{Bun}\, G)}} = \prod_{x \in X} \frac{q^{d \deg(x)}}{|G(\kappa(x))|}.$$

↓ Looks like BG fibered over X

We want to think of $\mathrm{Bun}_G(X)$ as a "continuous product" of BG's over X: let, for $x \in X$

$\mathrm{Bun}_G(x)(R)$ = groupoid of principal G bundles on $\mathrm{Spec}(R \otimes_{\mathbb{F}_q} \kappa(x))$. It has dimension $-d \cdot \deg(x)$.

Thus,

$$\frac{q^{d \deg(x)}}{|G(\kappa(x))|} = \frac{|\mathrm{Bun}_G(x)(\mathbb{F}_q)|}{q^{\dim \mathrm{Bun}_G(x)}}.$$

So our conjecture is

$$\frac{|\mathrm{Bun}_G(X)(\mathbb{F}_q)|}{q^{\dim \mathrm{Bun}_G(X)}} = \prod_{x \in X} \frac{|\mathrm{Bun}_G(x)(\mathbb{F}_q)|}{q^{\dim \mathrm{Bun}_G(x)}}.$$

We would like to count rational points by using the Grothendieck – Lefschetz trace formula.

Recall that if $Y/\mathbb{F}_q$ is a quasi-projective variety. Let $\overline{Y} = Y \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}$. Then $|Y(\mathbb{F}_q)|$ = # of fixed pts of geometric Frobenius $(=\varphi)$ which is given by:

$$Y(\mathbb{F}_q) = \sum_{i \geq 0} (-1)^i \, \mathrm{Tr}(\varphi \mid H^i_c(\overline{Y})).$$

Alternatively, using arithmetic as opposed to geometric Frob (which has small eigenvalues as opposed to geometric Frob which has large eigenvalues... the small eigenvalues will help convergence), and Poincaré Duality, we find:

$$\sum (-1)^i \operatorname{Tr}\left(\varphi^{-1} \mid H^i(\overline{Y}, \mathbb{Q}_\ell)\right) = \frac{|Y(\mathbb{F}_q)|}{q^{\dim Y}}.$$

So we may break up our conjecture into two pieces:

Conj. A:

$$\frac{|\mathrm{Bun}_G(X)(\mathbb{F}_q)|}{q^{\dim(\mathrm{Bun}_G(X))}} = \sum_{i\geq 0} (-1)^i \operatorname{Tr}\left(\varphi^{-1} \mid H^i(\overline{\mathrm{Bun}_G(X)} ; \mathbb{Q}_\ell)\right)$$

This is just the Grothendieck – Lefschetz Trace Formula for stacks.

Conj. B:

$$\sum_{i\geq 0} (-1)^i \operatorname{Tr}\left(\varphi^{-1} (\overline{\mathrm{Bun}_G(X)} ; \mathbb{Q}_\ell)\right) = \prod_{x\in X} \frac{q^{d \deg(x)}}{|G(\kappa(x))|}$$

Conjecture A is due to Kai Behrend. Two ingredients go into the proof: one uses the cobar spectral sequence to establish the G-L trace for quotient stacks X/G; then one stratifies $\mathrm{Bun}_G(X)$ by stacks of this form (using the Harder-Narasimhan filtration). Then one uses another spectral sequence

There is still an issue of convergence w/ $\Sigma (-1)^i \mathrm{Tr}(\mathrm{Fr}^*|H^i)$: it is usually **NOT** finite. Thankfully, this sum will converge absolutely.

Conjecture B will occupy most of our work this quarter.

VIII. Topological Inspiration.

Imagine we are studying $\mathrm{Bun}_G(X)$ where X is a compact Riemann surface. Now we may appeal to complex analysis. Recall the idea of Atiyah–Bott:

Every G-bundle $\mathcal{P} \longrightarrow X$ is a fiber bundle w/ simply connected fibers; since X has dimension 2, it is trivial in the category of smooth G-bundles. Thus, we may consider the trivial bundle as a smooth bundle and vary the holomorphic structure – i.e., the $\bar\partial$ structure. In particular, let $\mathcal{P}_{sm}$ be a smooth G-torsor on X. Then $T\mathcal{P}_{sm}$ is a G-equivariant vector bundle on $\mathcal{P}_{sm}$; we may write it as $\pi^*\mathcal{E}$, ($\mathcal{E}$ a bundle/X, $\pi: \mathcal{P}_{sm} \longrightarrow X$). We have an exact sequence:

$$0 \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{E} \longrightarrow T_X \longrightarrow 0.$$

A $\bar\partial$-connection on $\mathcal{P}_{sm}$ is a choice of complex structure on $\mathcal{E}$ such that the above is an exact sequence of complex vector bundles on X.

Let Ω denote this space. Ω is a torsor for the (∞-dim'l) vector space of $\mathbb{C}$ antilinear bundle maps $\mathrm{Hom}_{\overline{\mathbb{C}}}(T_X, \mathcal{E}_0)$; i.e. Ω is ∞-dim'l affine: thus Ω is contractible.

Prop. Let $\mathcal{G} = \mathrm{Aut}(\mathcal{P}_{sm})$ = all automorphisms of the smooth bundle $\mathcal{P}_{sm}$. Then

$$\mathrm{Bun}_G(X) \simeq \Omega / \mathcal{G}, \text{ and in particular}$$

$\mathrm{Bun}_G(X)$ has the homotopy type of $*/\mathcal{G} = B\mathcal{G}$.

Note: $\mathcal{G}$ (called the "Gauge Group") = space of all sections of $G \to X$.

Hope: describe $H^*(B\mathcal{G}, \mathbb{Q})$ in terms of $\{BG_x\}_{x \in X}$. There are three approaches:

1) Rational homotopy (expressing $H^*(B\mathcal{G}, \mathbb{Q})$ as cohomology of a differential graded Lie Algebra

2) $H^*(B\mathcal{G}, \mathbb{Q})$ via Factorization homology

3) $H_*(B\mathcal{G}, \mathbb{Q})$ via Non-Abelian Poincaré Duality.

All make extensive use of chain-level work: i.e., ∞-categories are needed.

1) If H is a path-connected group $H \times H \xrightarrow{m} H$ and $H \xrightarrow{\Delta} H \times H$ give $H_*(H, \mathbb{Q})$ the structure of a co-commutative

Hopf algebra. This can be done at the level of chains rather than homology.

Quillen gives a correspondence

$H \rightsquigarrow \mathfrak{g}(H)$, a differential graded Lie Algebra

such that

a) $\exists$ canonical iso

$$\mathbb{Q} \otimes \pi_* H \simeq H_* (\mathfrak{g}(H)),$$

Whitehead product on $\pi_{*+2} BH = \pi_* H$
$\rightsquigarrow [\, , \,]$ on $H_* (\mathfrak{g}(H))$.

b) $C_* (H, \mathbb{Q})$ (singular chain complex) is canonically quasi-isomorphic to universal enveloping $U(\mathfrak{g}(H))$, giving a Hopf alg. iso:

$$H_* (U(\mathfrak{g}(H))) \simeq H_* (H; \mathbb{Q}).$$

c) $\mathfrak{g}(H)$ is a complete invariant of rational homotopy type. $H^* (BH, \mathbb{Q})$ can be recovered as the Lie Algebra cohomology of $\mathfrak{g}(H)$.

Now: For all $U \subseteq X$ open we construct a presheaf

Let $\mathcal{G}_U$ = group of sections $G \times_X U \to U$.

$$\mathcal{F} : U \mapsto \mathfrak{g}(\mathcal{G}_U)$$

Letting $\mathcal{F}_n (U) = \mathfrak{g}(\mathcal{G}_U)_n$, and $\bar{\mathcal{F}}_n$ be sheafification. This gives a

Chain complex of presheaves:

$$\cdots \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F}_0 \to \mathcal{F}_{-1} \to \cdots$$

And so on sheaves:

$$\cdots \to \overline{\mathcal{F}}_2 \to \overline{\mathcal{F}}_1 \to \overline{\mathcal{F}}_0 \to \overline{\mathcal{F}}_{-1} \to \cdots$$

Thm.

$$\mathfrak{g}(\mathcal{G}) = \Gamma(X, \mathcal{F}_*) \to \Gamma(X, \overline{\mathcal{F}}_*) \to R\Gamma(X, \overline{\mathcal{F}}_*)$$

is a quasi-isomorphism of graded Lie Algebras.

I.e. cohomology of $\mathfrak{g}(\mathcal{G})$ is identified w/ hypercohomology of $\overline{\mathcal{F}}_*$.

Ex. Atiyah Bott. Say G is constant: $G = G_0 \times_{\mathbb{C}} X$.

Then $\overline{\mathcal{F}}_*$ is quasi-isomorphic to chain complex of constant sheaves w/ value $\mathfrak{g}(G_0)$. So:

$$\mathfrak{g}(\mathcal{G}) = C^*(X; \mathbb{Q}) \otimes_{\mathbb{Q}} \mathfrak{g}(G_0).$$

This leads us to discover that

$$H^*(\mathrm{Bun}_G(X), \mathbb{Q})$$ is isomorphic

to the graded symmetric algebra on

$$H_*(X, \mathbb{Q}) \otimes_{\mathbb{Q}} V^0[-1];$$ i.e. a tensor product

of a polynomial ring on $2r$ generators in even degree w/ exterior algebra on $2g r$ generators in odd degree.

2) Factorization Homology.

Atiyah–Bott showed that $\mathfrak{g}(\mathcal{G})$ is recovered as hypercohomology of the "local system" $x \mapsto \mathfrak{g}(G_x)$. This reflects the "continuous product of G_x over X" philosophy.

Want: local–global principal to compute

$$H^*(\mathrm{Bun}_G(C); \mathbb{Q}) \simeq H^*(B\mathcal{G}, \mathbb{Q}).$$

<u>Def.</u> For each $U \subseteq X$, let $\mathcal{B}(U) = C^*(B\mathcal{G}_U; \mathbb{Q})$.

$U \mapsto \mathcal{B}(U)$ gives covariant functor

$$\mathrm{Op}(X) \longrightarrow \mathrm{Chain}.$$

Let $\mathcal{U}$ = collection of open subsets that are disjoint union of disks. Let

$$\int \mathcal{B} = \mathrm{hocolim}\, \{\mathcal{B}(U)\}_{U \in \mathcal{U}} \quad \text{in } \mathrm{Chain}/\mathbb{Q}.$$

This is the factorization homology of $\mathcal{B}$.

Say $U = U_1 \amalg \dots \amalg U_n$, U_i disks. Set $x_i \in U_i$.

$\mathcal{G}_U$ is homeo to $\prod_{1 \le i \le n} \mathcal{G}_{U_i}$; evaluation at x_i gives homotopy equivalences

$$\mathcal{G}_{U_i} \longrightarrow G_{x_i}.$$

Thus

$$\bigotimes_{1 \le i \le n} C^*(BG_{x_i}, \mathbb{Q}) \xrightarrow{\sim} \bigotimes_{1 \le i \le n} C^*(B\mathcal{G}_{U_i}, \mathbb{Q})$$
$$\simeq C^*(B\mathcal{G}_U, \mathbb{Q})$$

$= B(U).$

So $\int B$ is a kind a cts tensor product

$$\bigotimes_{x\in X} C^*(BG_x; \mathbb{Q}).$$

<u>Thm.</u> (Local-to-Global principal for coh. of $Bun_G(X)$)

$$\int B \triangleq \operatorname{hocolim}_{U\in \mathcal{U}} B(U) \longrightarrow B(X) = C^*(B\mathcal{G}, \mathbb{Q}) = C^*(Bun_G(X), \mathbb{Q})$$

is a quasi-isomorphism. So cohomology of Bun_G can be identified w/ factorization homology of B.

3). Non-abelian Poincaré Duality.

For $U \subseteq X$, let $\mathcal{G}^c_U$ denote the subgroup of $\mathcal{G}$ consisting of those sects of P_{sm} which are identity outside compact subset of U. Let
$A(U)$ = chain complex $C_*(B\mathcal{G}^c_U, \mathbb{Q})$.
As before Taking $\mathcal{U}$ = opens iso to disj. union of disks define

$$\int A \triangleq \operatorname{hocolim}\{A(U)\}_{U\in\mathcal{U}}.$$

Call this the Factorization homology of A.

<u>Ex.</u> Say $U \subseteq X$ is an open disk, $\ni x \in X$.

$U \times_X G$ is diffeo to $U \times G_x$, so that $\mathcal{G}^c_U$ is identified w/ compactly supported maps from U into G_x.

$U \simeq \mathbb{R}^2$ gives equivalence of

$\mathcal{G}^c_U$ w/ $\Omega^2(G_x)$ (double loop space); we have:

$B\mathcal{G}^c_U \simeq \Omega^2(BG_x) \simeq \Omega(G_x)$. Now: $U_i \ni x_i$ as before.

$$\bigotimes_{1 \le i \le n} C_*(\Omega^2 BG_{x_i}; \mathbb{Q}) \simeq \bigotimes_{1 \le i \le n} C_*(B\mathcal{G}^c_{U_i}; \mathbb{Q})$$

$$\simeq C_*(B\mathcal{G}^c_U; \mathbb{Q})$$

$$= \mathcal{A}(U).$$

I.e. terms in $\{\mathcal{A}(U)\}_{U \in \mathcal{U}}$ can be identified

w/ tensors $\bigotimes_{x \in S} C_*(\Omega^2 BG_x; \mathbb{Q})$, $S \subseteq X$ finite.

Thus $\int \mathcal{A}$ is a kind of the tensor product

$$\bigotimes_{x \in X} C_*(\Omega^2(BG_x), \mathbb{Q}).$$

Moreover: $\Omega^2(BG_x) = \Omega(G_x)$ is homotopy eq. to $G(K_x)/G(O_x)$

the affine Grassmannian ... which has an algebro-geometric interpretation.

Thm. (Nonabelian Poincare Duality)

$$\int \mathcal{A} = \operatorname{hocolim}_{U \in \mathcal{U}} \mathcal{A}(U) \longrightarrow \mathcal{A}(X) = C_*(B\mathcal{G}, \mathbb{Q}) = C_*(\mathrm{Bun}_G(X), \mathbb{Q})$$

is a quasi-isomorphism.

Approaches I and II and II and III are related by various kinds of Koszul Duality.

Our task is to find the Algebro-Geometric Analogue.

We will formulate Nonabelian Poincaré Duality, which depends upon the following amazing fact:

The space $G(K)$ of rational functions is contractible.

We will then use this to apply the analogue of approach 1, w/ the Lie algebra $\mathfrak{g}_*(K)$.

The action of Frob on this complex

∞-categories are necessary as we will be working w/ complexes.